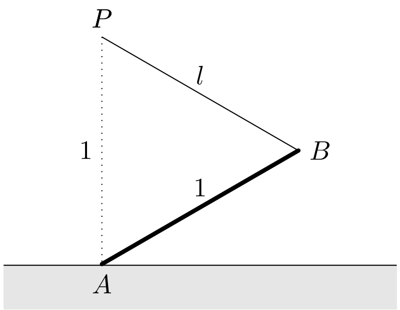
3601. Determine which of $(0, 0)$ and $(-0.8, 1.675)$ lies closer to the hyperbola $xy = 1$.

3602. A uniform rod AB of mass m and length 1 m is in equilibrium. Point A rests on rough horizontal ground, and point B is attached, by a light string of length l, to a point P, which is 1 m vertically above A.



Show that the tension is given by $T = \frac{1}{2}mgl$.

3603. A geometric sequence is given by $u_n = 2^n$, for $1 \leq n \leq 10$. Two different terms are chosen at random from the sequence.

(a) Find the probability that the two terms have a sum greater than 1000.

(b) Find the probability that the two terms have a sum less than 15.

3604. A function g, expressed as an improper algebraic fraction, has instruction

$$\text{g}(x) = \frac{9x^2}{3x+1}.$$

(a) Write this improper fraction as the sum of a polynomial and a proper fraction.

(b) Hence, integrate g.

3605. Using graphs, or otherwise, prove that the linear approximation to $\arccos x$, for small x measured in radians, is

$$\arccos x \approx \tfrac{\pi}{2} - x.$$

3606. A fair coin is tossed four times, and the number n of tosses in the longest run of either heads or tails is recorded. This process is repeated many times. Find the average value of n.

3607. The cubic $y = x^3 - x^2$ is rotated by a right-angle clockwise around the origin. Find the equation of the transformed graph.

3608. State, with a reason, whether the following are valid implications:

(a) $a > b \implies \dfrac{1}{a} < \dfrac{1}{b}$,

(b) $|a| > |b| \implies \dfrac{1}{a} < \dfrac{1}{b}$,

(c) $|a| > |b| \implies \dfrac{1}{|a|} < \dfrac{1}{|b|}$.

3609. By integration, prove the area formula for a circle.

3610. Solve the equation $x\sqrt{x} - 6x + 12\sqrt{x} - 8 = 0$.

3611. Find the maximum value of $x^2 - y^2$, subject to the condition that $4y = x^2$.

3612. It is given that $y = \text{g}(x)$ has rotational symmetry around the point $(a, 0)$, and also that

$$\int_0^a \text{g}(x)\,dx = b.$$

Find the following integrals, in terms of a, b:

(a) $\displaystyle\int_a^{2a} 1 + \text{g}(x)\,dx$,

(b) $\displaystyle\int_0^{2a} 1 + \text{g}(x)\,dx$.

3613. In electrophoretic coating, deposits appear on a cathode. The rate at which this happens, in grams per second, is modelled, t seconds after a current is switched on, by

$$\frac{dm}{dt} = te^{p - \frac{1}{q}t},$$

where p and q are positive constants.

(a) Show that peak rate of deposition is

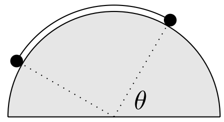
$$\frac{dm}{dt} = qe^{p-1}.$$

(b) Explain why the total mass deposited, over the whole process, is given by

$$m_{\text{max}} = \int_0^\infty te^{p - \frac{1}{q}t}\,dt.$$

(c) Find the total mass deposited.

3614. Particles with masses m and $2m$ are in equilibrium, resting on the smooth surface of a half-cylinder of radius r. They are attached to one another by a light, inextensible string of length $\frac{1}{2}\pi r$.



(a) Draw force diagrams for both particles.

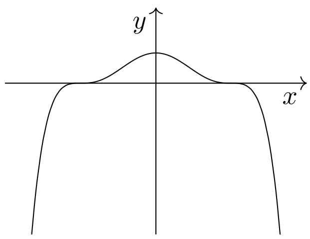
(b) Determine the value of $\tan\theta$.

3615. Prove that $\dfrac{d(\operatorname{cosec}\theta)}{d\theta} \equiv \dfrac{2\cos\theta}{\cos 2\theta - 1}$.

3616. A computer simulation randomly chooses a value of N from the set $\{1, 2, ..., 10\}$, and then randomly chooses a value X from the set $\{1, 2, ..., N\}$. Find

(a) $\mathbb{P}(X \geq 9)$,
(b) $\mathbb{P}(N = 10 \mid X \geq 9)$.

3617. Determine whether the equation $y = \left(1 - x^2\right)^3$ could produce the following graph:



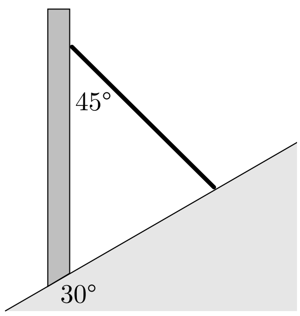
3618. Show that a triangle with side lengths $(33, 34, 35)$ cm will not fit inside a circle of radius 19 cm.

3619. An area is to be calculated by $\displaystyle\int_{-1}^{1} x^3 - x \, dx$.

(a) Show, by calculation, that direct integration and the trapezium rule with four strips yield precisely the same result.
(b) Explain, in terms of the symmetry of $x^3 - x$, why this is the case.

3620. Prove that $\dfrac{d(e^x)}{d(\ln x)} \equiv xe^x$.

3621. A ladder of mass m is leaning against and at 45° to a smooth vertical wall. It stands on a rough slope, which runs down towards the wall, of inclination 30°. The coefficient of friction is μ. The foot of the ladder is on the point of slipping down the slope, towards the base of the wall.



(a) Draw a force diagram for the ladder.
(b) Give equations for
   i. vertical equilibrium,
   ii. horizontal equilibrium,
   iii. moments around the foot of the ladder.
(c) Show that $\mu = \dfrac{5\sqrt{3} - 8}{11}$.

3622. Two curves and a straight line are given by

$$2y = x^2 + 2,$$
$$2x = y^2 + 2,$$
$$2x + 2y + 1 = 0.$$

Show that these loci are equidistant.

3623. An integral I is defined as

$$I = \int \frac{2}{x^2 - 1} \, dx.$$

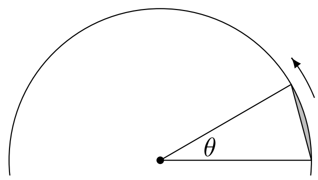
(a) Show that e^I may be expressed as $A \left| \dfrac{x-1}{x+1} \right|$.
(b) Give any conditions on the constant A.

3624. A statistician sets up and carries out a two-tailed hypothesis test on a sample of bivariate data (x, y), testing for correlation. The test reveals insufficient evidence for correlation.

A second statistician thinks the test should have been one-tailed, however, and conducts a new test, using the same data.

State, with a reason, whether it is possible for the two tests to yield different results.

3625. In a circle of fixed radius r, a segment subtends a variable angle θ at the centre. θ is increasing at a rate of 1 radian per second.



Show that, when θ is small, the rate of change of the area of the segment can be approximated by

$$\frac{dA}{dt} \approx \tfrac{1}{4} r^2 \theta^2.$$

3626. A curve is defined by the equation

$$x = \frac{25}{16 + y^2}.$$

A tangent is drawn at the point with $y = 4$. Show that this tangent re-intersects the curve on the x axis.

3627. A student writes the following, when solving for roots $\theta \in [0, 360°)$ in a trig equation:

$$\tan(2\theta + 15°) = 1$$
$$\implies 2\theta + 15° = 45°$$
$$\implies 2\theta = 30°$$
$$\therefore \quad \theta \in \{15°, 195°\}$$

Explain the error, and correct it.

3628. Sketch $y = \dfrac{x+1}{x^2+x+1}$.

3629. Ten books, including a trilogy, are arranged on a shelf. Show that there are 241920 different ways of arranging the books such that the three books of the trilogy are all next to each other.

3630. Express the fraction $^{164}/_{6283}$ in the form $^1/_p + {}^1/_q$, where p and q are prime numbers. You are given the prime factorisations

$$\begin{aligned} 164 &= 2^2 \cdot 41, \\ 6283 &= 61 \cdot 103. \end{aligned}$$

3631. This question concerns the evaluation, using a technique called *telescoping*, of the sum

$$\sum_{r=1}^{n} \frac{1}{r(r+1)}.$$

(a) Express $\dfrac{1}{r(r+1)}$ in partial fractions.
(b) Show that, when written in partial fractions, most of the terms of the sum cancel out.
(c) Hence, show that $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)} \equiv \frac{n}{n+1}$.

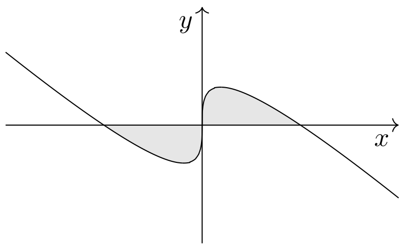
3632. Function $\mathrm{f}_n(x)$ is defined on the domain $[0, 1]$ as

$$\mathrm{f}_n : x \mapsto k_n\big(x^{n-1} - x^n\big).$$

The constant k_n is to be chosen such that the area beneath the graph $y = \mathrm{f}_n(x)$ over $[0, 1]$ is 1.

(a) Sketch, on a single pair of axes, the curves $y = \mathrm{f}_2(x)$ and $y = \mathrm{f}_4(x)$.
(b) Show that $k_n = n(n + 1)$.

3633. The curve $(x + y)^3 - x = 0$ intersects the x axis three times, enclosing two regions.



Show that each enclosed region has area $\frac{1}{4}$.

3634. Factorise $48x^3 - 340x^2 + 802x - 630$ fully.

3635. A graph is defined implicitly by the equation

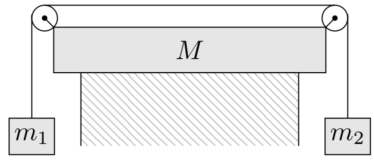
$$xy(x + y) = 2.$$

Show that this graph may be expressed as

$$y = -\frac{x}{2} \pm \sqrt{\frac{x^2}{4} + \frac{2}{x}}.$$

3636. Find the sum of the integers from 1 to 100 which are not divisible by either 4 or 5.

3637. A pulley system of two blocks of mass $m_1 < m_2$ is set up on a large block of mass M, which rests on a fixed tabletop. The large block is free to move horizontally over the tabletop. Friction is neglected throughout, and the string is modelled as light and inextensible.



(a) Explain carefully whether, when the system is released, the large block of mass M will move horizontally.
(b) Show that the string accelerates at

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2}.$$

3638. Show that there are no points of inflection on

$$y = \frac{1-x}{1+x}.$$

3639. The variables X_1 and X_2 are both distributed as $B(3, 0.5)$, independently of one another.

(a) Find $\mathbb{P}(X_1 + X_2 = 3)$.
(b) You are given that $\mathbb{P}(X_1 + X_2 = 3)$ is equal to $\mathbb{P}(Y = 3)$, where $Y \sim B(n, {}^1/_2)$. Write down the value of n.

3640. Use integration by parts with $u = x^3$ to determine

$$\int x^5 \sqrt{x^3+1}\, dx.$$

3641. Solve $2 \sin t - 2\sqrt{3} \cos t = \sqrt{6} + \sqrt{2}$ for $t \in [0, 2\pi]$.

3642. A colony of bacteria, population size P, is growing exponentially with time t. Some values are given below:

t	12	36	60
P	576	2304	a

Determine the value a.

3643. The following *sum-product identities* are given:

$$\begin{aligned} \sin x + \sin y &\equiv 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right), \\ \cos x + \cos y &\equiv 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right). \end{aligned}$$

Prove the following identity:

$$\tan(\theta + \phi) \equiv \frac{\sin 2\theta + \sin 2\phi}{\cos 2\theta + \cos 2\phi}.$$

3644. The velocity of a particle is modelled by

$$v = \begin{cases} 3t+8, & t \in [0,4) \\ -2t^2+16t-12, & t \in [4,\infty) \end{cases}$$

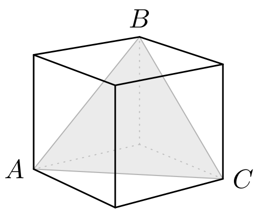
(a) Find the maximum velocity.

(b) Show that the particle does not return to its original position in the first ten seconds.

3645. Solve the equation $\dfrac{(\ln x)^2 + 12(\ln 2)^2}{\ln x} = \ln 128.$

3646. You are given that the volume of a pyramid is $V = \frac{1}{3}Ah$, where A is the area of the base and h is the perpendicular height.

The diagram shows a cube of unit side length, with a triangle ABC formed by three vertices. $\triangle ABC$ divides the interior of the cube into two regions.



Two points are chosen at random inside the cube. The probability that these lie in different regions is denoted p. Show that $p = 5/18$.

3647. A function is defined over $\mathbb{R}$ as

$$\mathrm{f}(x) = \begin{cases} \dfrac{x}{e^x - 1}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

It is given that

$$\int_0^\infty \mathrm{f}(x) = \frac{\pi^2}{6}.$$

(a) With reference to this integrand, explain how it is possible for an integral to give a finite value over an infinite domain.

(b) The integral is approximated using a four-strip trapezium rule on the interval $[0,4]$. Show that the error is around 3%.

3648. A chord is drawn on the curve $y = x^4 - 4x^2 - 5x$, joining the points with x coordinate ± 2. Show that this chord is also a tangent to the curve.

3649. Find the probability that two numbers chosen at random on the interval $[0,1]$ differ by less than $\frac{1}{3}$.

3650. Prove that, for any constant a, the tangent to the following curve at $x = a$ passes through the origin:

$$y = \frac{a^2x^2}{a^2+x^2}.$$

3651. The locus of points satisfying $\sin y = \sin 2x$ is a tiling of the plane by quadrilaterals. Determine the area of each of these quadrilaterals.

3652. Either prove or disprove the following statement: "Given a set of $n+1$ distinct linear equations in n unknowns, there cannot be a unique solution."

3653. A planar shape is transforming. Its perimeter is

$$P = 4\sqrt{x^2+y^2},$$

where x and y are variable lengths in cm.

(a) Differentiate this formula with respect to time, simplifying your answer.

(b) When $x = 3$ and $y = 4$, the rates of increase of x and y, in cm/s, are 2 and 1 respectively. Find the rate of change of the perimeter at this moment.

3654. Find $\displaystyle\lim_{h\to 0} \frac{\sqrt[3]{x^3+h} - x}{h}$, for $x \neq 0$.

3655. Three couples sit down at random around a round table. Show that the probability that exactly one couple is sitting together is $\frac{2}{5}$.

3656. A theorem of topology says that, if the surface of a sphere is divided into F faces by E edges and V vertices, then $V - E + F = 2$. On a football, the stitched pieces of fabric form such a division. One common pattern uses pentagonal and hexagonal pieces, with three faces meeting at every vertex.

(a) Show that $V = \frac{1}{3}(5p+6h)$, where p and h are the numbers of pentagons and hexagons.

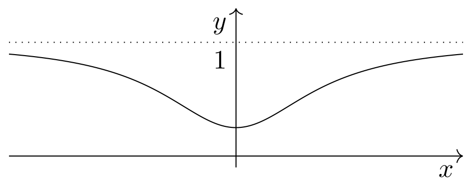
(b) Find equivalent formulae for E and F in terms of p and h.

(c) Hence, prove that every such division must have precisely 12 pentagons.

3657. A function is given, over the real numbers, by

$$\mathrm{g} : x \longmapsto \frac{x^2+1}{x^2+4}.$$

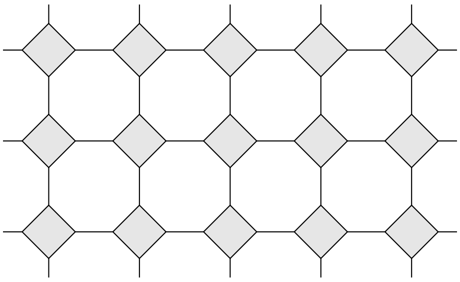
The graph of $y = \mathrm{g}(x)$ is shown below.



Find the inverse of g on the largest possible real interval $[k,\infty)$. Define the domain, codomain and algebraic instruction in your answer.

3658. Prove that, if $y = \mathrm{f}'(x)$ has a line of symmetry at $x = a$, then $\mathrm{f}(a + x) + \mathrm{f}(a - x)$ is constant.

3659. The floor of a large room is covered in small tiles. These are regular octagons and squares, forming the tessellating pattern shown:



A small ball is dropped onto the floor. Show that the the probability that it lands in one of the squares is approximately 17%.

3660. For non-zero constants a, b and n, two parametric equations are given as

$$\begin{aligned} ax + by &= \sin nt, \\ bx - ay &= \cos nt. \end{aligned}$$

Show that, when the parameter takes values $t \in \mathbb{R}$, these equations define a circle in the (x, y) plane.

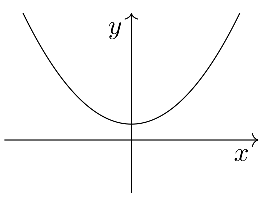
3661. Find the equations of the two asymptotes of the curve $y = \ln(1 + e^x)$, and hence sketch it.

3662. By differentiating $x = \tan y$, show that

$$\frac{d}{dx}(\arctan x) \equiv \frac{1}{x^2 + 1}.$$

3663. Solve the equation $x^{0.7} + x^{0.4} = 72x^{0.1}$.

3664. The graph below is of $y = \mathrm{h}''(x)$, for some quartic function h defined over $\mathbb{R}$.



Show that h can have at most two real roots.

3665. A function is given as $\mathrm{f}(x) = \displaystyle\sum_{r=0}^{\infty} x^r$.

(a) Determine the largest real domain over which this function is well defined.

(b) Find the range of f over this domain.

3666. Sketch and shade the regions of the (x, y) plane which satisfy the inequality

$$\left(x^2 + 4y^2 - 4\right)\left(4x^2 + y^2 - 4\right) \leq 0.$$

3667. The area A under the graph $y = x^2$ between $x = 0$ and $x = 1$ can be approximated with n rectangles of width $\frac{1}{n}$. Placing the top-right corner of each rectangle on the curve gives an over-estimate of the area:

$$A_{\text{over}} = \sum_{r=1}^{n} \frac{r^2}{n^3}.$$

(a) By placing the top-left corners on the curve, obtain an under-estimate A_{under}.

(b) The sum of the first n squared integers is

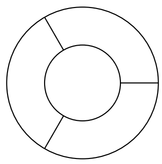
$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

Using this result, show that

$$\frac{(n-1)\left(n - \frac{1}{2}\right)}{n^2} < 3A < \frac{(n+1)\left(n + \frac{1}{2}\right)}{n^2}.$$

(c) Hence, by letting $n \to \infty$, determine A.

3668. The regions of the following diagram are randomly coloured, each red, orange, yellow, green or blue. Multiple regions can be the same colour.



Find the probability that no two regions end up coloured the same.

3669. A workman of mass m is carrying a heavy load of mass M. Determine, in terms of m and M, the average frictional force between the ground and the workman's feet when he is moving, at constant speed,

(a) along level ground,

(b) up a ramp of inclination 30°.

3670. Show that $\displaystyle\int_{-1}^{0} \frac{x^2 + x}{x - 1}\, dx = \tfrac{3}{2} - \ln 4$.

3671. Sketch the relation $x^4y^2 - 10 = 9x^2y$.

3672. A curve has equation

$$y = \frac{100}{x^2 + 9}.$$

A line crossing the x axis at $57/8$ is tangent to the curve at point P, at which $x = p$. Determine all possible values of p.

3673. Find the periods of the following functions, defined in radians:

(a) $x \longmapsto \tan \frac{1}{2}x$,

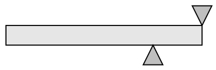
(b) $x \longmapsto \tan \frac{1}{2}x + \operatorname{cosec} 2x$.

3674. You are given that x and y satisfy

$$e^{x+y} + 1 = e^x + e^y.$$

Show that at least one of x or y must be zero.

3675. A makeshift shelf of mass m, made from a uniform piece of wood, is supported as below, shown in side view. The lower support divides the shelf in the ratio $k : 1$, where $k > 1$.



Show that the total magnitude of the reaction forces exerted by the supports is kmg.

3676. Consider graph G, defined by the equation

$$y = \frac{\sin x + 1}{\sin x + 2}.$$

(a) Show that G has no vertical asymptotes.

(b) Show that G oscillates between local minima at $y = 0$ and local maxima at $y = 2/3$.

(c) You are given that the second derivative is

$$\frac{d^2y}{dx^2} = -\frac{4 \sin x + \cos 2x + 3}{2(\sin x + 2)^3}.$$

Show that G turns more sharply at the minima than at the maxima.

(d) Hence, sketch G.

3677. Three different pentagonal faces of a dodecahedron are selected at random. Find the probability that the three faces surround a single vertex.

3678. Show, by considering approach from the positive and negative directions, that the following limit is not well defined:

$$\lim_{x \to 2} \frac{x^2 - 2x}{|x - 2|}.$$

3679. (a) Solve the inequality $y^3 - y < 0$.

(b) Sketch $\sqrt{x} = y^3 - y$.

3680. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning polynomial functions f and g:

① $\mathrm{f}''(x) \equiv \mathrm{g}''(x)$,

② $\mathrm{f}'(x) \equiv \mathrm{g}'(x)$.

3681. Show that, for small values of x,

$$\frac{(1 + 2\sin x)^7}{(1 + 3x)^4} \approx 1 + 2x + 6x^2.$$

3682. Functions f and g are both defined over $\mathbb{R}$, and both have range $[-1, 1]$. State, with a reason, whether the following are necessarily true:

(a) $\mathrm{f} + \mathrm{g}$ has range $[-2, 2]$ over $\mathbb{R}$,

(b) $\mathrm{f}(x) + \mathrm{g}(x) \in [-2, 2]$ for all $x \in \mathbb{R}$.

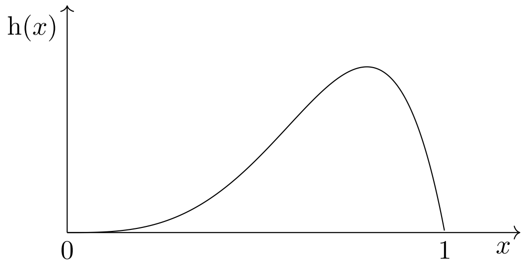
3683. Two curves are given as $px^2 + qy^2 = 1$ and $y = x^2$, where p and q are non-zero constants. Show that there are values of p and q for which the curves do not intersect.

3684. A particle is oscillating with coordinates at time t given by $(\sin 3t, \cos 4t)$. Show that the particle momentarily comes to rest during the oscillation.

3685. Data dumps, each of a million numbers $x_i \in [0, 1]$, emerge from a network. The distribution of these numbers (an idealised histogram) is modelled by the function

$$\mathrm{h}(x) = k\big(x^3 - x^7\big),$$

where the area beneath $\mathrm{h}(x)$ represents frequency.



(a) Show that $k = 8 \times 10^6$.

(b) Find the modal value x.

(c) Determine the median value, according to the model, of the numbers in the data dumps. Give your answer in exact form.

3686. An isosceles triangle has area 360 and perimeter 100. Find its edge lengths, given that they are integers.

3687. Two inequalities are given as follows:

$$xy \le 2,$$
$$|x - y| \le 1.$$

(a) Write down the boundary equations and find their four intersections.

(b) Shade the region R of the (x, y) plane whose points satisfy both inequalities simultaneously.

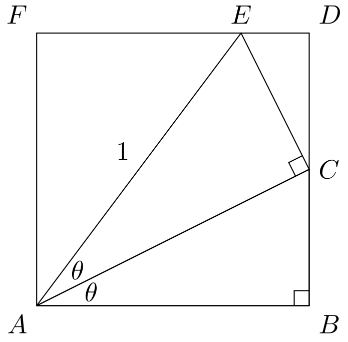
(c) Find the exact area of region R, giving your answer in the form $a + \ln b$, where $a, b \in \mathbb{N}$.

3688. A graph is defined, for $x \in [0, 1]$, by

$$y = \sqrt{1 - x^n}.$$

Show that, if $n \in (0, 1)$, then the tangent to the graph is parallel to the y axis at $x = 0$ and $x = 1$.

3689. Use the diagram below to prove the double angle formula $\sin 2\theta \equiv 2 \sin\theta \cos\theta$.



3690. A function f has algebraic instruction $x \mapsto x \ln x$.

(a) Explain why any evaluation of $\text{f}(x)$ at $x = 0$ requires a consideration of limits.

(b) Show that f is decreasing on $(0, 1/e)$.

(c) Hence, prove that $\lim_{x \to 0^+} \text{f}(x)$ does not diverge.

3691. The position x and acceleration a of an object are linked by the relationship

$$6x + a = 0.$$

Verify that $x = c \sin\omega t + d \cos\omega t$, for any constants $c, d \in \mathbb{R}$, satisfies this relationship, where $\omega > 0$ is a constant to be determined.

3692. Determine the largest real domain and codomain over which $\text{f}(x) = \ln(x - 1) - \ln x$ is invertible.

3693. A long rectangular strip of paper, of width l, is to be used to wrap a cylinder of circumference C. The strip mustn't overlap itself, nor leave gaps. Determine, in terms of l and C, the angle θ away from the circumference at which the strip must be placed.

3694. A curve is given by $y = (x^2 - x - 1)e^x$.

(a) Find any axis intercepts.

(b) Find and classify all stationary points.

(c) Determine the behaviour as $x \to \pm\infty$.

(d) Hence, sketch the curve.

3695. Disprove the following statement: "When regions are marked on a 2D map, it is possible, using only three colours, to shade the regions so that no two regions sharing a border are the same colour."

3696. True or false?

(a) $y = \dfrac{x}{x^2 + 1}$ has a horizontal asymptote,

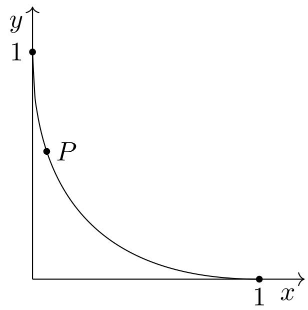
(b) $y = \dfrac{x^2}{x^2 + 1}$ has a horizontal asymptote,

(c) $y = \dfrac{x^3}{x^2 + 1}$ has a horizontal asymptote.

3697. Show that, for $n \in \mathbb{N}$, $\displaystyle\int_0^\pi \sin^2(nx)\,dx = \tfrac{\pi}{2}$.

3698. In a game, a standard die is rolled. The number showing is noted, and the die is rolled again that many times. The score is the total number of sixes across all throws (potentially 7). Show that the probability of obtaining 4 is approximately 1%.

3699. The curve $\sqrt{x} + \sqrt{y} = 1$ is shown below.



(a) Verify that a point P on the curve may be given parametrically as $(\cos^4 t, \sin^4 t)$.

(b) Show that $\dfrac{dy}{dx} = -\tan^2 t$.

(c) Hence, find the equation of the tangent to the curve at $(1/16, 9/16)$, giving your answer in the form $ax + by = c$, where $a, b, c \in \mathbb{N}$.

3700. A rigid object is in equilibrium under the action of three forces in an (x, y) plane:

$$a\mathbf{i} - 5\mathbf{j} \text{ N at } (4, 2),$$
$$3\mathbf{i} + b\mathbf{j} \text{ N at } (1, 0),$$
$$\mathbf{i} - 4\mathbf{j} \text{ N at } (-1, c).$$

Determine a, b, c.

— End of 37th Hundred —